An Introduction to Fuzzy Sets Theory

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Outline

1. Introduction
2. Fuzzy sets

Introduction

From: *B. Kosko, Scientific American 1993*

- **Computers do not reason as brain do.** Computers "reason" when they manipulate precise facts that have been reduced to strings of zeros and ones and statements that are either true or false. The human brain can reason with vague assertions or claims that involve uncertainties or value judgments: "The air is cool", or "That speed is fast" or "She is young". Unlike computers, humans have common sense that enables them to reason in a world where things are inly partially true.
- Fuzzy logic is a branch of machine intelligence that helps computers paint gray, commonsense pictures of an uncertain world. Logicians in the 1920s first broached its key concept: everything is a matter of degree.

Introduction

From: *B. Kosko, Scientific American 1993*

- Fuzzy logic manipulates such vague concepts as "warm" or 'still dirty' and so helps engineers to build air conditioners, washing machines and other devices that judge how fast they should operate or shift from one setting to another even when the criteria for making those changes are hard to define.
- When mathematicians lack specific algorithms that dictate how a system should respond to inputs, fuzzy logic can control or describe the system by using "common sense' rules that refer to indefinite quantities.

Introduction

From: *B. Kosko, Scientific American 1993*

- No known mathematical model can back up a truck-and-trailer rig from a parking lot to a loading dock when the vehicle starts from a random spot. Both humans and fuzzy systems can perform this nonlinear guidance task by using practical but imprecise rules such as "If the trailer turns a little to the left, then turn it a little to the right."
- Fuzzy systems often glean their rules from experts. When no expert gives the rules, adaptive fuzzy systems learn the rules by observing how people regulate real systems.

Introduction

Applications of Fuzzy Sets Theory (from somewhere??)

Machine Systems	**Human-Based Systems**	**Human/Machine Systems**
picture/voice recognition Chinese character recognition natural language understanding intelligent robots crop recognition process control production management car/train opration safety/maintenance systems breakdown diagnosis electric power systems operations fuzzy controllers home electrical appliance control automatic operation	human reliability model cognitive psycology thinking/beaviour models sensory investigations public awareness investigation risk assesment enviromental assesment human relation structures demand trend models social pychology category analysis	medical diagnosis inspection data processing trnafusion consultation expert systems CAI CAE optimization planning personnel management development planning equipement diagnostics quality evaluation insurance systems human interfaces management decision-making multiporpouse decision-making knowledege bases data bases

Introduction

From: *B. Kosko, Scientific American 1993*

- A recent wave of commercial fuzzy products, most of them from Japan, has popularized fuzzy logic.
- In 1980 the contracting firm of F. L Smidth & Company in Copenhagen first used fuzzy system to oversee the operation of a Cement kiln.
- In 1983 Hitaci turned over control of a subway in Sendai, Japan. to a fuzzy system.
- Since then, Japanese companies have used fuzzy logic to direct hundreds of household appliances and electronics products.

Introduction

From: *B. Kosko, Scientific American 1993*

- Applications for fuzzy logic extend beyond control systems. Recent theorems show that in principle fuzzy logic can he used to model any continuous system, be it based in engineering or physics or biology or economics.
- Investigators in many fields may find that fuzzy, commonsense models are more useful or accurate than are standard mathematical ones.

Introduction
A brief hystory

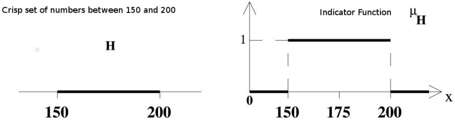
- 1910-1913 Bertrand Russell & Alfred North Whitehead: "Principia Mathematica"
- 1920 Jan Łukasiewicz (Polish) multi-modal logic: paper "On Three-valued Logic"
- 1937 Max Black - vague sets: paper "Vagueness: An exercise in logical analysis"
- 1965 Lotfali (Lofti) Askar Zadeh (born in Azerbaijan in 1921): paper "Fuzzy sets" (Information and Control. 1965; 8: 338–353). Zadeh applies the Łukasiewicz logic to each element of a set and proposes a complete algebra for fuzzy sets.

Fuzzy Sets

Crisp sets

- Fuzzy sets are sets whose elements have degrees of membership.
- Fuzzy sets were introduced by Lotfi A. Zadeh (1965) as an extension of the classical notion of set.
- In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set.

Fuzzy Sets
Example



- In classical sets theory, the set **H** of real numbers from 150 to 200 is:

$$H = \{r \in \Re \mid 150 \leq r \leq 200\}$$

- The indicator function (or characteristic function) $\mu_H(r)$ gives the membership of each element of the universe $\Re$ to h:

$$\mu_H(r) = \begin{cases} 1 & \text{if } 150 \leq r \leq 200 \\ 0 & \text{otherwise} \end{cases}$$

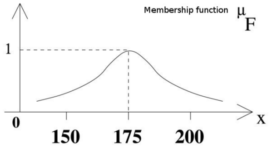
Fuzzy Sets
Crisp sets

- Fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$.
- Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.
- Classical bivalent sets are in fuzzy set theory usually called crisp sets.
- The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.

Fuzzy Sets
Example

- Let we define the fuzzy sets F of real numbers that are **close to 175** by meas the following membership function:

$$\mu_F(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(r-175)^2}$$



Fuzzy Sets

Definition (Fuzzy set)

A fuzzy set A in X is a set of ordered pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$

μ_A is called the membership function, $\mu_A : X \to M$, where M is the membership space where each element of X is mapped to.

- If $M = \{0, 1\}$, A is a *crisp* set.

Fuzzy Sets

Membership functions and probabilities

Questions About Fuzzy Sets : 

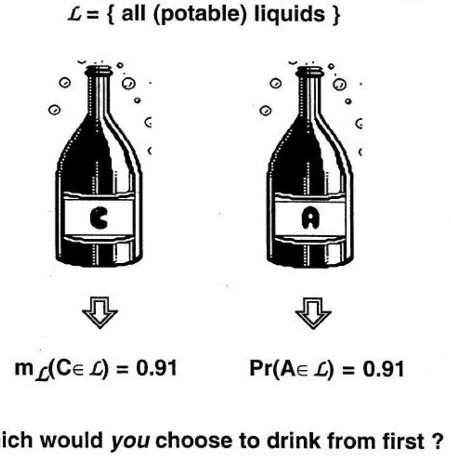
Q. Fuzzy sets are a clever disguise for *Probability* ?

A. Nope !! ------------------ *Philosophically* Different

Fuzzy Sets

Membership functions and probabilities

Suppose you had been in the desert for a week without a drink and you came upon two bottles

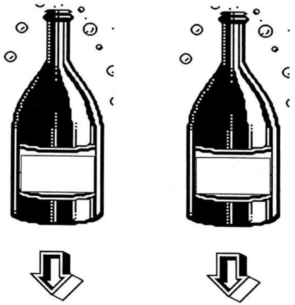


Fuzzy Sets
Membership functions and probabilities

- C could contain, say, swamp water. That is membership of 0.91 means that the contents of C are fairly similar to perfectly potable liquids (e.g., pure water).
- The probability that A is potable = 0.91 means that over a long run of experiments, the contents of A are expected to be potable in about 91% of the trials; in the other 9% the contents will be hydrochloric acid.

Fuzzy Sets
Membership functions and probabilities

And *after observation* of C and A ?



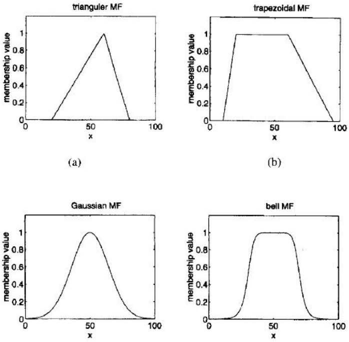
$m_{\mathcal{L}}(C) = 0.91$ $\quad$ $Pr\ (A \in \mathcal{L}) = 0$

Fuzzy Sets
Membership functions and probabilities

- *While it is of great intellectual interest to establish the proper connections between FL and probability, this author does not believe that doing so will change the ways in which we solve problems, because both probability and FL should be in the arsenal of tools used by engineers* [Mendel, 1995].

Fuzzy Sets
Memberships

Defined by the user, as it is need by the problem to model



Clustering methods and fuzzy sets

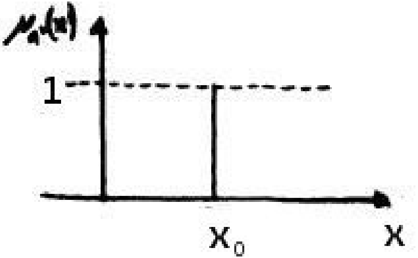
- Jim Bezdek (1981) introduced the concept of hard and fuzzy partition in order to extend the notion of membership of pattern to clusters.
- The motivation of this extension is related to the fact that a pattern often cannot be thought of as belonging to a single cluster only. In many cases, a description in which the membership of a pattern is shared among clusters is necessary.

Fuzzy Sets
Fuzzy singleton

Definition (Fuzzy singleton)

The singleton is a fuzzy set A associated to a crisp number x_0. Its membership function is

$$\mu_A(x) = \begin{cases} 1 & \textit{if } x = x_0 \\ 0 & \textit{otherwise} \end{cases}$$



Fuzzy Sets
Example 1

A real estate agent wants to classify the houses offering to customers.
Let $X = \{1, 2, 3, ..., 10\}$ the number of bedrooms. The fuzzy set A *"kind of comfortable home for a family of 4"* can be defined as

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.3)\}.$$

Fuzzy Sets
Example 2

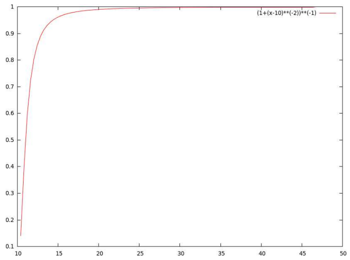
$\tilde{A}$ = "real numbers considerably larger than 10"

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

where

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq 10 \\ (1 + (x - 10)^{-2})^{-1} & x > 10. \end{cases}$$

```
gnuplot> set xrange [10:50]
gnuplot> plot (1+(x-10)**(-2))**(-1)
```



Fuzzy Sets
Notation

$$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 + \cdots = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i \quad \text{if } X \text{ discrete}$$

$$\tilde{A} = \int_X \mu_{\tilde{A}}(x)/x \quad \text{if } X \text{ continuous}$$

Note: in this context, the meaning of the symbols $+$ $\sum$ $\int$ is union of elements

Fuzzy Sets
Normal fuzzy set

Definition (Normal fuzzy set)

$\tilde{A}$ normal $\iff sup_x \mu_{\tilde{A}}(x) = 1$

NOTE: if $\tilde{A}$ is not normal, it can be normalized:

$$\mu_{\tilde{A}}(x) \longrightarrow \frac{\mu_{\tilde{A}}(x)}{sup_x \mu_{\tilde{A}}(x)}$$

Fuzzy Sets
Example 3

$\tilde{A}$ = "integers close to 10"

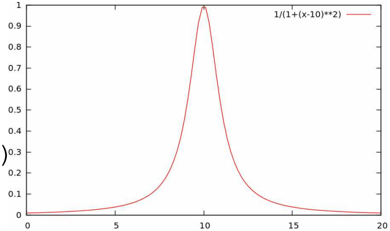
$$\tilde{A} = .1/7 + .5/8 + .8/9 + 1/10 + .8/11 + .5/12 + .1/13$$

Fuzzy Sets
Example 4

$\tilde{A}$ = "real numbers close to 10"

$$\tilde{A} = \int_{\mathcal{R}} \frac{1}{1+(x-10)^2} \Big/ x$$

```
gnuplot> set xrange [0:20]
gnuplot> plot 1/(1+(x-10)**2)
```



Fuzzy Sets

Definition (Support of a fuzzy set $S(\tilde{A})$)

The support $S(\tilde{A})$ of a fuzzy set is a crisp set containing all elements of $S(\tilde{A})$ with $\mu_A(x) > 0$,i.e.,

$$S(A) \;=\; \{\; x \;\mid \mu_A(x) > 0, x \in X\}.$$

Definition (α-level set (or α-cut))

The α-level set A_α of the fuzzy set A is a crisp set containing the elements of A with membership degree at least α, i.e.,

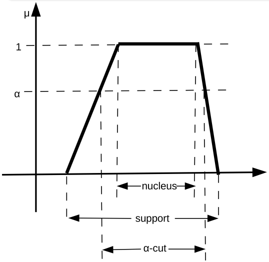
$$A_\alpha \;=\; \{\; x \;\mid \mu_A(x) \geq \alpha, x \in X\}.$$

Fuzzy Sets

Definition (*Strong* α-level set (or Strong α-cut))

The strong α-level set A_α of the fuzzy set A is a crisp set containing the elements of A with membership degree greater than α, i.e.,

$$\acute{A}_\alpha = \{\, x \mid \mu_A(x) > \alpha, x \in X\}.$$



A_0 support ; A_1 nucleus

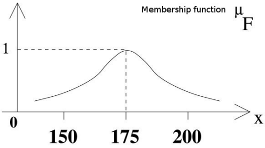
Fuzzy Sets
Fuzzy number

Definition (Fuzzy number)

A fuzzy number F in a continuous universe U , e.g., a real line, is a fuzzy set F in U which is normal and convex

- Example:

$$\mu_F(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(r-175)^2},$$

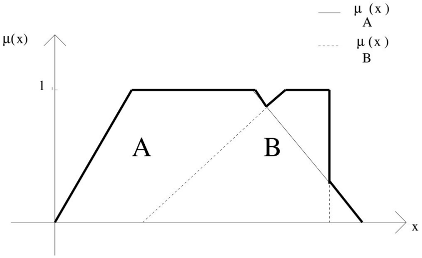


Fuzzy Sets
Algebra of fuzzy sets

Definition (Union of fuzzy sets)

Let A e B be two fuzzy sets in X. The union of A and B is the fuzzy set $D = A \cup B$ with membership function

$$\mu_D(x) = max\{\mu_A(x), \mu_B(x)\} \ \forall x \in X.$$

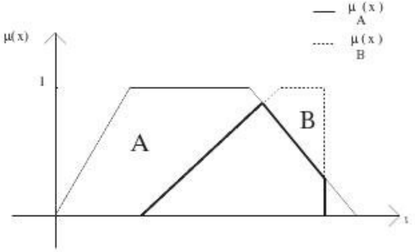


Fuzzy Sets
Algebra of fuzzy sets

Definition (Intersection of fuzzy sets)

Let A e B be two fuzzy sets in X. The intersection of A and B is the fuzzy set $D = A \cap B$ with membership function

$$\mu_D(x) \; = \; min\{\mu_A(x), \mu_B(x)\} \;\; \forall x \in X.$$

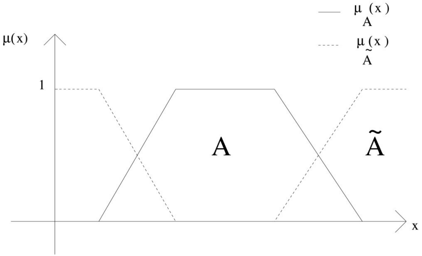


Fuzzy Sets
Algebra of fuzzy sets

Definition (Complement of a fuzzy set)

Let A be a fuzzy set in X. The complement of A in X is the fuzzy set Ã with membership function

$$\mu_{\tilde{A}}(x) = 1 - \mu_A(x) \ \forall x \in X.$$



Fuzzy Sets

Triangolar norms, t-norms (FUZZY AND)

intersection	$x \wedge y = \min\{x, y\}$
algebraic product	$x \cdot y = xy$
bounded product	$x \odot y = \max\{0, x + y - 1\}$
drastic product	$x \cap y = \begin{cases} x & y = 1 \\ y & x = 1 \\ 0 & x, y < 1. \end{cases}$

Fuzzy Sets
Triangolar co-norms (FUZZY OR)

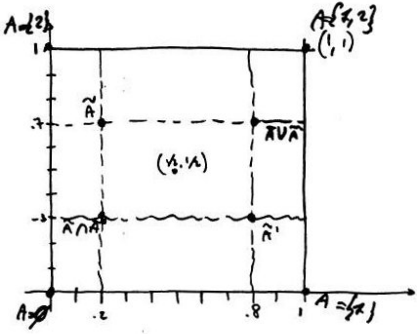
union	$x \vee y = \max\{x, y\}$
algebraic sum	$x \dotplus y = x + y - xy$
bounded sum	$x \oplus y = \min\{1, x + y\}$
drastic sum	$x \,\dot\cup\, y = \begin{cases} x & y = 0 \\ y & x = 0 \\ 1 & x, y > 0 \end{cases}$
disjoint sum	$x \Delta y = \max\{\min(x, 1 - y), \min(1 - x, y)\}.$

Fuzzy Sets

Geometrical interpretation [Kosko, 1991]

- *Fuzzy hypercube*

$$\tilde{A} = \{(1, .2), (2, .7)\}$$



- Principle of Non-Contradiction: $A \cap A' = \emptyset$
- Principle of Excluded Middle: $A \cup A' = U$

Fuzzy Sets

Geometrical interpretation [Kosko, 1991]

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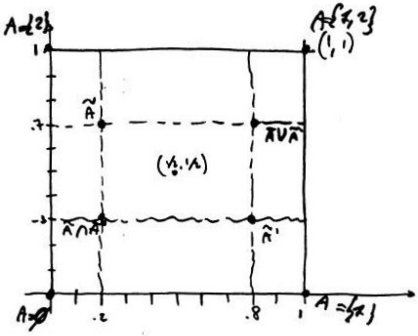

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Fuzzy Sets

Geometrical interpretation [Kosko, 1991]

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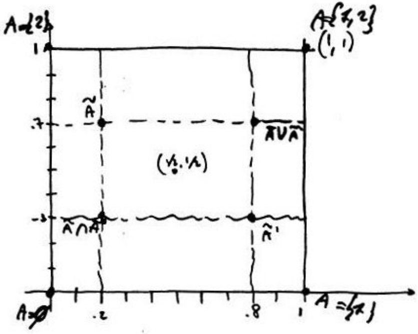


- Principle of Non-Contradiction: $A \cap A' = \emptyset$
- Principle of Excluded Middle: $A \cup A' = U$

Aggregation operations

- Multicriteria
- Multi-expert
- Information Fusion

Aggregation operations

- Several fuzzy sets are combined in a desirable way to produce a single fuzzy set.
- An *aggregation operation* on n fuzzy sets ($n \geq 2$) is defined as a function
  $h : [0, 1]^n \rightarrow [0, 1]$.
  When applied to fuzzy sets $A_1, A_2, \ldots, A_n$ defined on X, function h produces an aggregate fuzzy set A by operating on the membership grades of these sets for each $x \in X$. Thus,
  $\mu_A(x) = h(\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)) \quad \forall x \in X$.

Aggregation operations

Axioms expressing the essence of the notion of aggregation:

1. $h(0, 0, \ldots, 0) = 0$ and $h(1, 1, \ldots, 1) = 1$ (*boundary condition*).
2. $\forall$ pair of $< a_1, a_2, \ldots, a_n >$ and $< b_1, b_2, \ldots, b_n >$ of n-ples such that $a_i, b_i \in [0, 1] \forall i \in \mathbf{N}_n$, if $a_i \leq b_i \forall i \in \mathbf{N}_n$, then $h(a_1, a_2, \ldots, a_n) \leq h(b_1, b_2, \ldots, b_n)$;
   i.e., h is *monotonic increasing* in all its arguments.
3. h is a *continuous* function.

Aggregation operations

Additional axioms (context depending):

1. h is a *symmetric* function in all its arguments, i.e.,
   $h(a_1, a_2, \ldots, a_n) = h(a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)})$
   for any permutation p on $\mathbf{N}_n$.
2. h is a *idempotent* function, i.e.,
   $h(a, a, \ldots, a) = a$
   $\forall a \in [0, 1]$.

Aggregation operations

Valid aggregation operations:

- Fuzzy intersections and unions
- Generalized means

  $$h_\alpha(a_1, a_2, \ldots, a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n} \right)^{1/\alpha}$$

  - $\alpha = -1$: *harmonic mean*
    $h_{-1}(a_1, a_2, \ldots, a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$
  - $\alpha = 1$: *aritmentic mean*
    $h_1(a_1, a_2, \ldots, a_n) = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$

Aggregation operations

Valid aggregation operations:

- Ordered Weighted Averanging (OWA) operations [Yager, 1988]
  - *weighting vector*
    $\mathbf{w} = < w_1, w_2, \ldots, w_n > \quad w_i \in [0,1] \quad \forall i \in \mathbf{N}_n$ and
    $$\sum_{i=1}^{n} w_i = 1$$

Aggregation operations

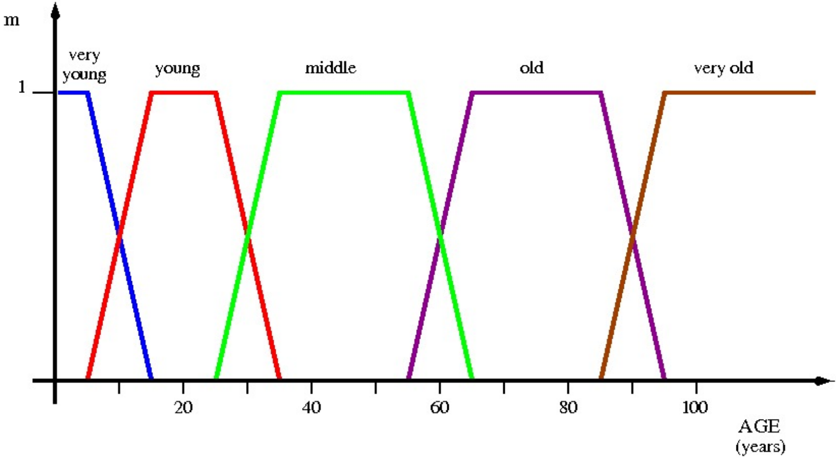
- Ordered Weighted Averanging (OWA) operations [Yager, 1988]
  - the OWA operation associated with **w** is the function:
    $h_{\mathbf{w}}(a_1, a_2, \ldots, a_n) = w_1 b_1 + w_2 b_2 + \ldots + w_n b_n$
    where b_i for any $i \in \mathbf{N}_n$ is the i-th largest element in $a_1, a_2, \ldots, a_n$.
    Vector $< b_1, b_1, \ldots, b_n >$ is a permutation vector of vector $< a_1, a_2, \ldots, a_n >$ in wich the elements are ordered: $b_i \geq b_j$ for any pair $i, j \in \mathbf{N}_n$
  - Example: **w** $=< .3, .1, .2, .4 >$
    $h_{\mathbf{w}}(.6, .9, .2, .7) = .3 \times .9 + .1 \times .7 + .2 \times .6 + .4 \times .2 = .54$

Aggregation operations

NOTE: Ordered Weighted Averanging (OWA) operations [Yager, 1988]

- $\mathbf{w} =< 1/n, 1/n, \ldots, 1/n >\rightarrow h_{\mathbf{w}}$ arithmetic mean
- lower bound
  $\mathbf{w}_\star =< 0, 0, \ldots, 1 >\rightarrow h_{\mathbf{w}_\star} = min(a_1, a_2, \ldots, a_n)$
- upper bound
  $\mathbf{w}^\star =< 0, 0, \ldots, 1 >\rightarrow h_{\mathbf{w}^\star} = max(a_1, a_2, \ldots, a_n)$

Fuzzy Sets
Linguistic variable



Fuzzy Sets
Linguistic variable

Definition (Linguistic variable)

A linguistic variable is characterized by a quintuple $(x, U, T(x), G, M)$ in which

- x is the name of variable;
- U is the universe of discourse;
- $T(x)$ is the term set of x , that is, the set of names of linguistic values of x with each value being a fuzzy number defined on U;
- G is a syntactic rule for generating the names of values of x;
- M is a semantic rule for associating each value with its meaning.

Fuzzy Sets
Linguistic variable and biomedical knowledge

- $age = \{very\ young, young, middle, old, very\ old\}$
- $blood\ glucose\ level = \{slightly\ increased, increased, significantly\ increased, strongly\ increased\}$.
- $insulin\ doses = \{none, low, medium, high\}$