A PRIMER ON TOPOLOGY

Lecture 1

WHAT IS A SURFACE?

The outside part or uppermost layer of something [New Oxford American Dictionary]

A plane or curved two-dimensional locus of points (as the boundary of a three-dimensional region) [Merriam-Webster Dictionary]

A two-dimensional topological manifold [Wikipedia]

TOPOLOGY

A.k.a. ***Rubber-sheet geometry***

It studies invariance of geometric objects upon elastic deformations:

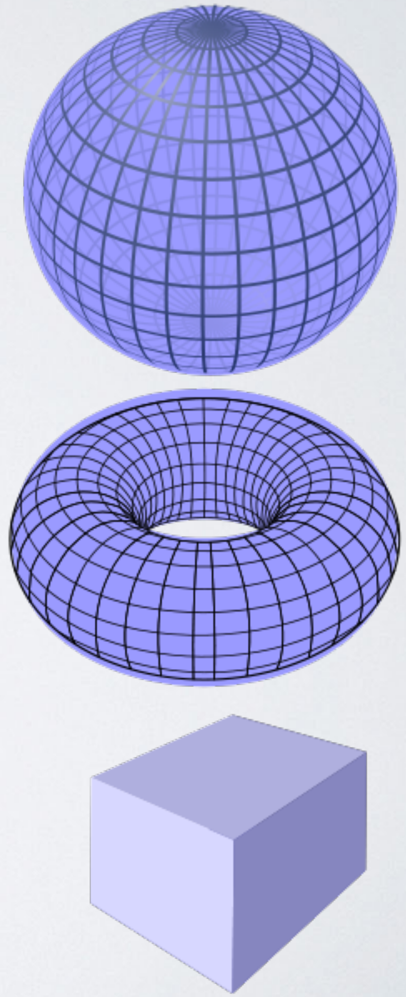
two shapes are the same iff they can be mapped into each other through a bijective and bi-continuous mapping (***homeomorphism***)

TOPOLOGY

Examples:

1.Sphere and cube can be mapped into each other by stretching/pulling/pushing: they are homeomorphic

2.Sphere and torus cannot be mapped into each other without tearing or puncturing: they are *not* homeomorphic



TOPOLOGICAL SPACE

A topology for a set X is a family T of subsets of X satisfying the following properties:

1. X and the empty set $\emptyset$ belong to T
2. union of any collection of elements of T is in T
3. intersection of any finite collection of elements of T is in T.

The elements of T are called ***open sets***.

Topological space: a pair (X, T), where X is a set and T is a topology for X

METRIC SPACE

A *metric* (or *distance function*) on a set X is a function $d : X \times X \rightarrow \mathbf{R}$ *such that:*

1. $d(x,y) \geq 0$
2. $d(x,y) = 0$ if and only if $x = y$
3. $d(x,y) = d(y,x)$ (symmetry)
4. $d(x,y) + d(y, z) \geq d(x, z)$ (triangle inequality)

A *metric space* is a pair (X,d) such that d is a metric on X

METRIC TOPOLOGY

- ***Ball (spherical neighborhood):*** $B(x,r) = \{y \in X \mid d(x,y) < r\}$
- Spherical neighborhoods for all $x \in X$ and $r \in \mathbf{R}^+$ form a base for a topology of X, called the *metric topology*
- Open sets in the metric space: all sets that can be obtained as union of (possibly infinitely many) disks

EUCLIDEAN SPACE

- ***n-dimensional Euclidean space:*** $E^n = (R^n, d)$ where

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

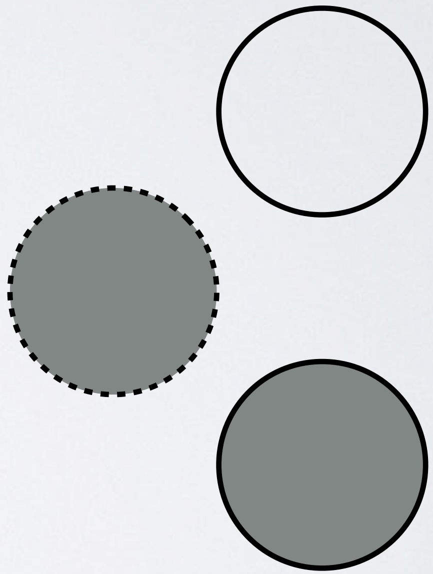
- the topology induced by the Euclidean distance is called Euclidean topology

- in this course: $n = 1, 2, 3$

EUCLIDEAN SPACE

Let $p \in \mathbf{E}^n$, $r \in \mathbf{R}^+$:

- Sphere: $S^{n-1}(p) = \{ x \in \mathbf{E}^n \mid d(x,p) = r \}$
- Ball: $B^n(p) = \{ x \in \mathbf{E}^n \mid d(x,p) < r \}$
- Disk: $D^n(p) = \{ x \in \mathbf{E}^n \mid d(x,p) \leq r \}$
- Remark: $D^n(p) = B^n(p) \cup S^{n-1}(p)$



OTHER BASIC CONCEPTS

(X, T) topological space

- ***Closed set:*** $A \subseteq X$ is *closed* in T iff $X - A$ is open in T
- ***Neighborhood:*** $V \subseteq X$ is a *neighborhood* of $p \in X$ iff $\exists\, U \subseteq V$ open set s.t. $p \in U$
- ***Boundary:*** p is a *boundary point* of $A \subseteq X$ iff every neighborhood of p intersects both A and $X - A$
- The *boundary* ∂A of A is the set of all boundary points of A
- Prop.: a set is closed iff it contains all points of its boundary

OTHER BASIC TOPOLOGICAL CONCEPTS

- ***Interior:*** p is an *interior point* of $A \subseteq X$ iff $\exists\, U \subseteq A$, U neighborhood of p.

- The *interior* $i(A)$ of A is the largest open set contained in A
- The *closure* $c(A)$ is the smallest closed subset of X that contains A
- $i(A) = A - \partial(A)$
- $c(A) = A \cup \partial(A)$

HOMEOMORPHISM

A *map* $f: (X,T) \rightarrow (X',T')$ is *continuous* iff

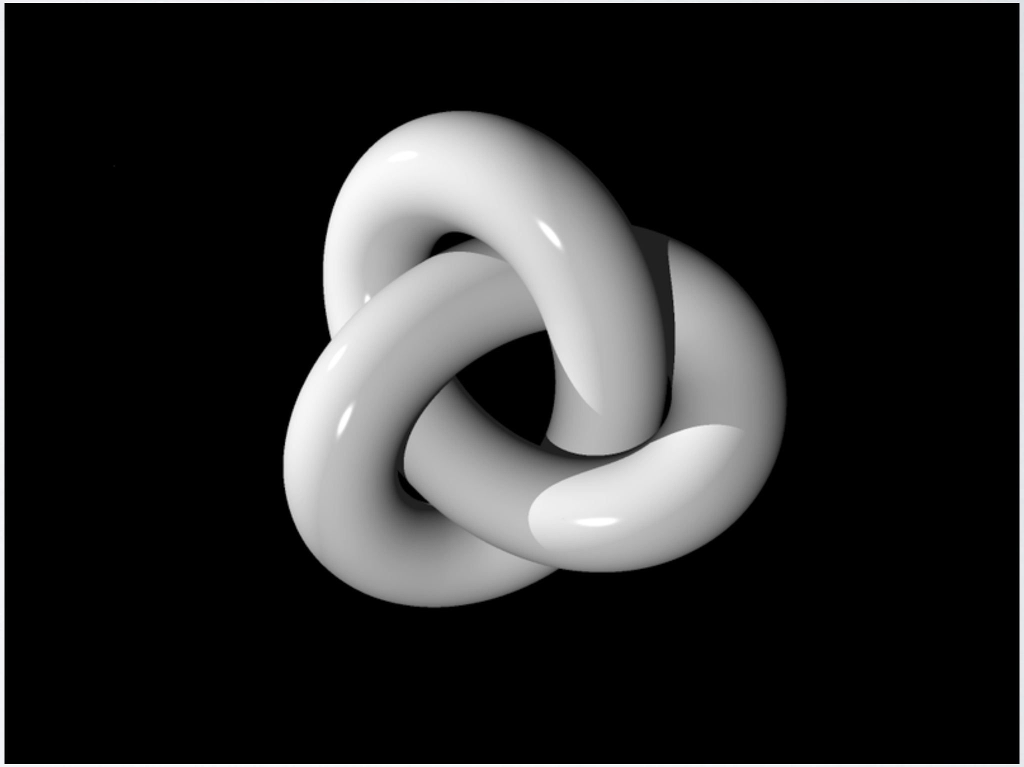
for each open set U in X', $f^{-1}(U)$ is an open set in X

A ***homeomorphism*** is a bijective map f such that both f and f^{-1} are continuous. Two spaces (X,T) and (X',T') are *homeomorphic* iff there exist a homeomorphism $f: (X,T) \rightarrow (X',T')$

Homeomorphisms partition the class of topological spaces into equivalence classes

Any property preserved by homeomorphisms is called a *topological invariant*

HOMEOMORPHISM



CONNECTEDNESS

A topological space X is *connected* if X cannot be written as the union of two non-empty disjoint open subsets of X

A topological space X is *path-connected* if and only if, for every pair $x,y \in X$, there exists a continuous map $\alpha: [0,1] \to X$ such that $\alpha(0)=x$ and $\alpha(1)=y$. The map α is called a path from x to y

Prop.: a path-connected space is connected

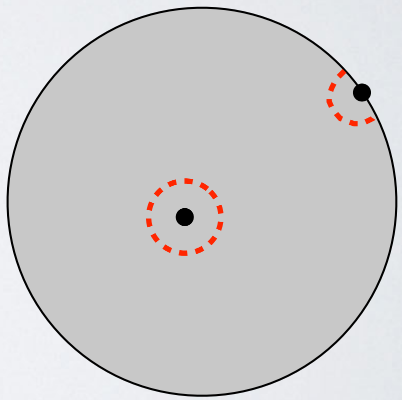
A connected (path-connected) component of X is a maximal connected (path-connected) subset of X

We will consider only path-connected subsets of Euclidean space $\mathbf{E}^n$

MANIFOLDS

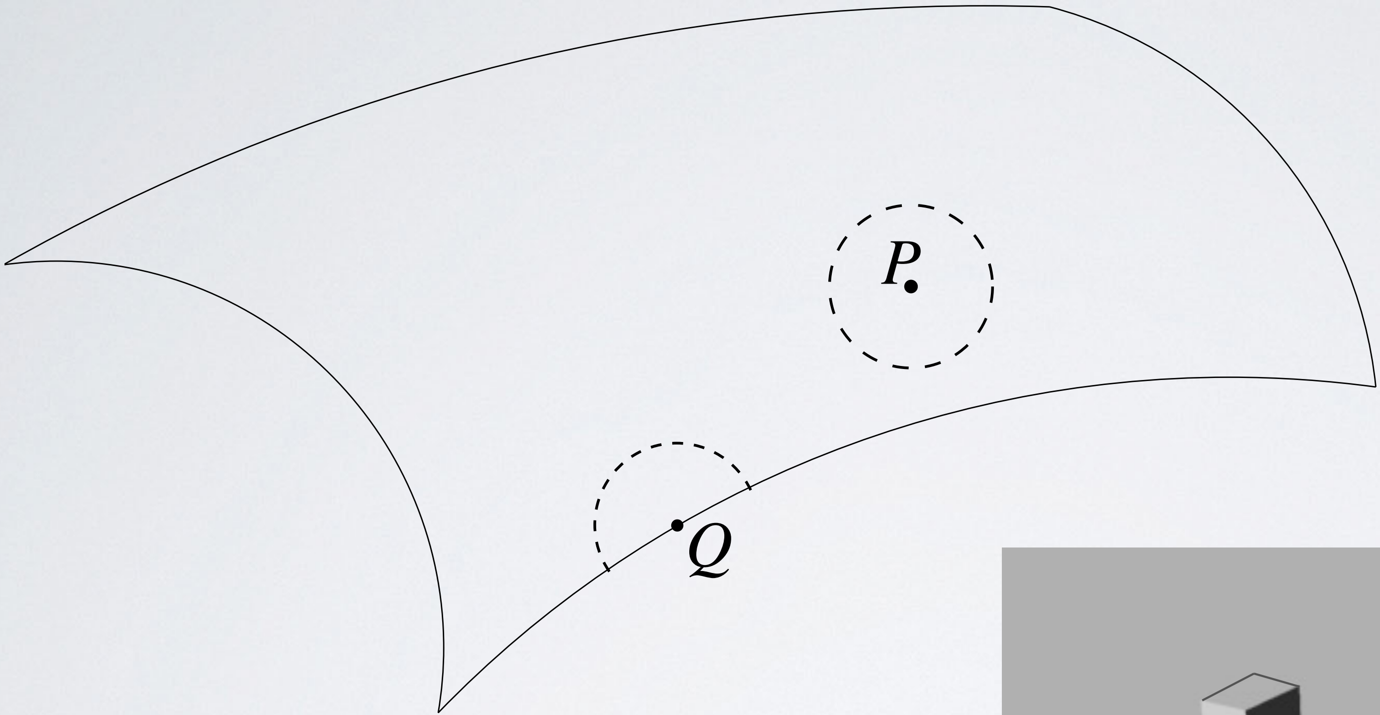
A sub-space *M* of the Euclidean space $\mathbf{E}^n$ is called a *d-manifold* ($d \leq n$) if and only if every point *p* of *M* has a neighborhood in *M* homeomorphic to the unit *d*-dimensional ball B^d

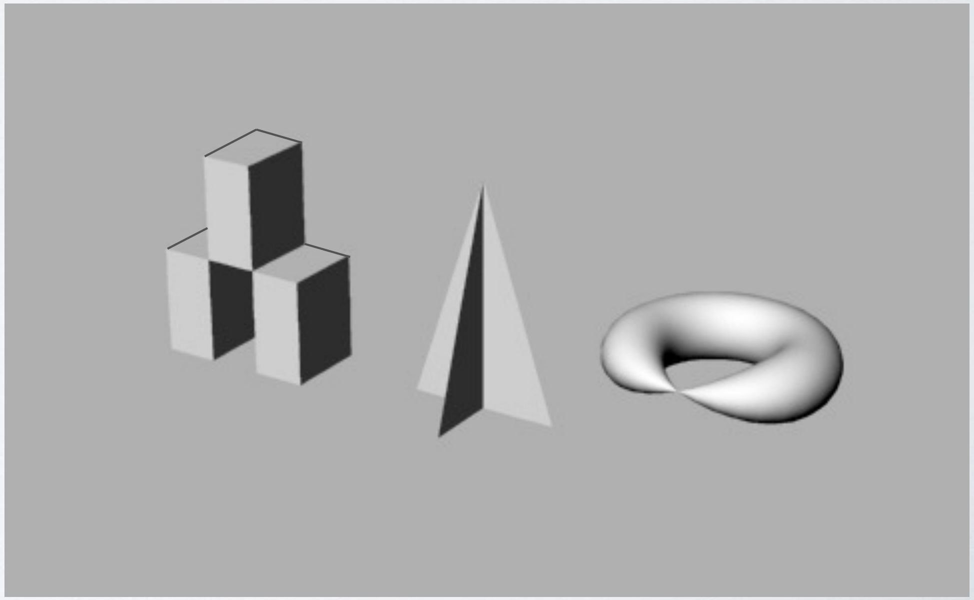
A sub-space *M* of the Euclidean space $\mathbf{E}^n$ is called a *d-manifold with boundary* ($d \leq n$) if and only if every point *p* of *M* has a neighborhood in *M* homeomorphic either to the unit *d*-dimensional ball B^d or to the closed half-plane H^d



The boundary of a manifold with boundary is formed by all those points that have a neighborhood homeomorphic to the closed half-plane

MANIFOLDS





Non-manifold objects

MANIFOLDS

The boundary of a d-manifold with boundary is a $(d-1)$-manifold (without boundary)

The disk D^d is a d-manifold with boundary

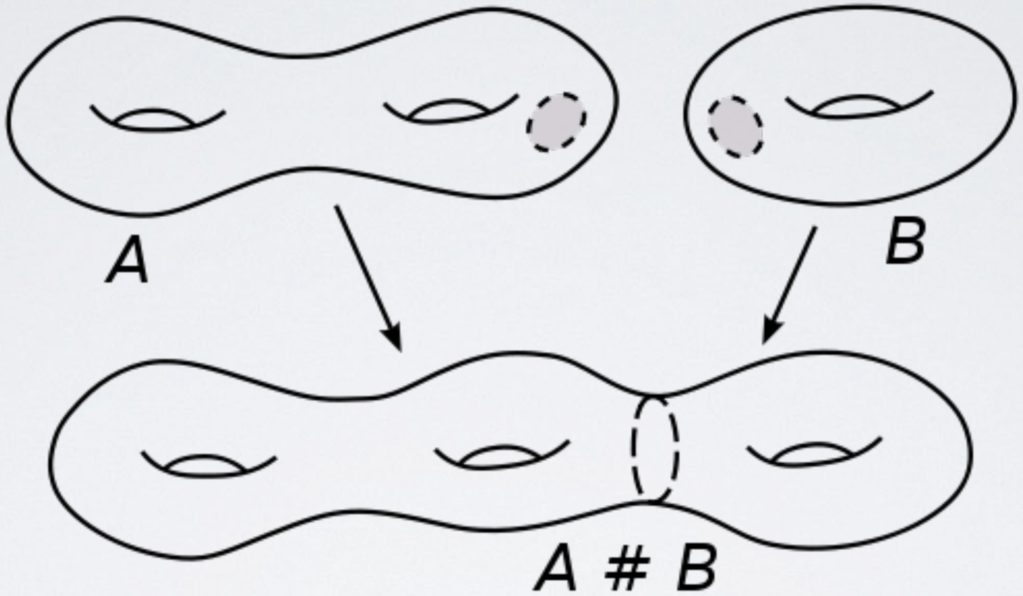
The sphere S^{d-1} is a $(d-1)$-manifold (without boundary)

SURFACES

A *surface* is a 2-manifold (either with or without boundary)

A surface with boundary is bordered by one or more *loops*: each loop is homeomorphic to the 1-sphere S^1

SURFACES



Surfaces can be combined by *connected sum*:

remove one disk from each surface and glue them at the borders of such disks

CLASSIFICATION OF SURFACES

Every surface without boundary is homeomorphic to some member of one of these families:

1. the sphere

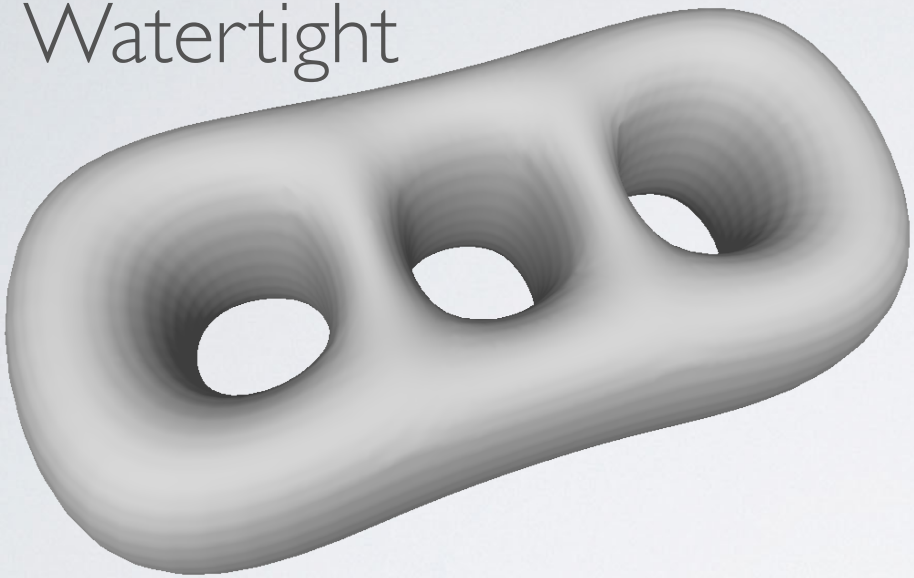
2. the connected sum of g tori, for $g \geq 1$ (a doughnut with g holes)

3. the connected sum of k real projective planes, for $k \geq 1$

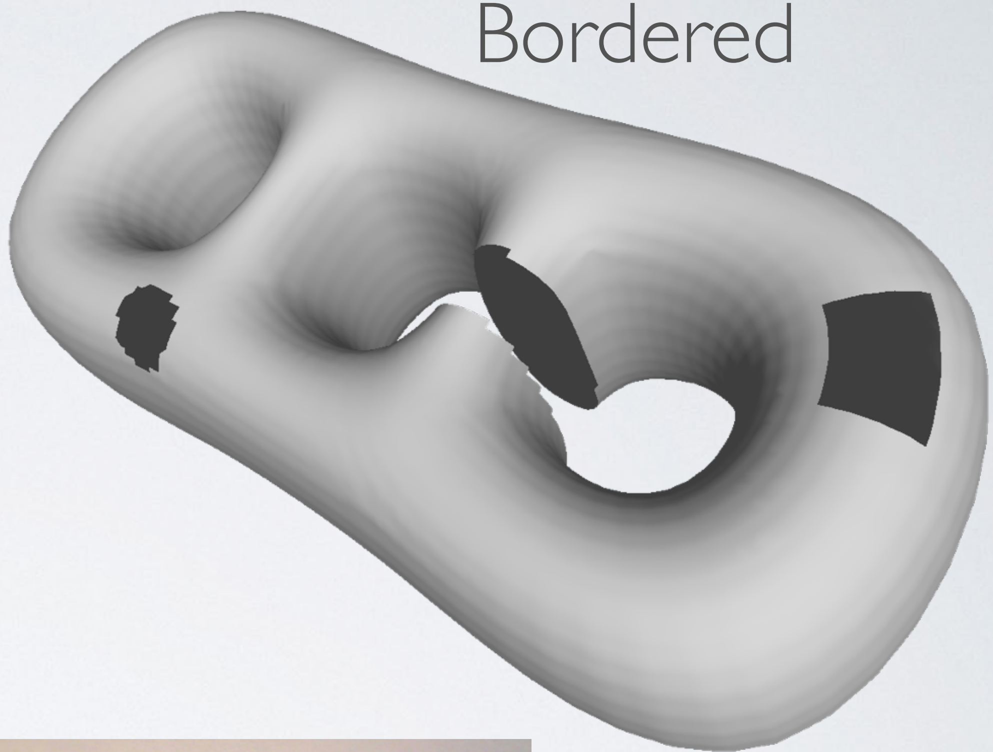
We will consider only members of the first two classes (*orientable surfaces*): an orientable surface without boundary is *watertight*

SURFACES

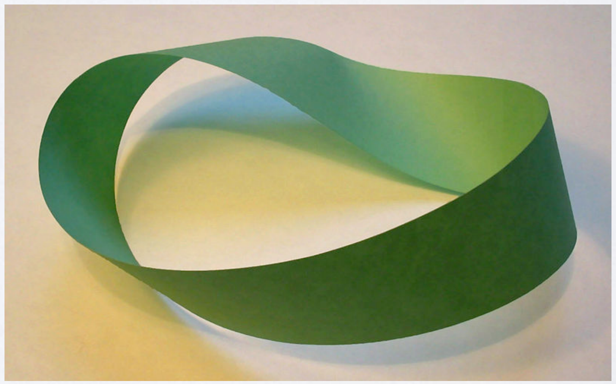
Watertight



Bordered



The Moebius strip:
a non-orientable surface
obtained by cutting a disk
from the projective plane



SUMMARY

- We have revised basic topological concepts: open and closed sets, boundary, interior, hoeomorphism, ...
- Surfaces are locally similar to "pieces" of the 2D plane
- They can either have borders or not
- They can be classified easily with respect to known families, depending on their number of handles and holes