Computer Aided Verification
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Temporal Logic

Temporal Logic: A Class of Modal Logics

- Modal Logic: alternative notions of truth like *is it possible/necessary that φ is true?*
- Temporal logic is a special type of modal logic in which the truth of a formula depends on the time in which it is evaluated
- Typical temporal operators are
  - Eventually Φ: in some future instant Φ is true
  - Always Φ: in all future instants Φ is true

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- LTL and CTL are incomparable logics: There exist formulas in one logic that are not expressible in the other
- LTL and CTL are submsumed by CTL*, which in turn, is subsumed by the μ-calculus (a fixpoint logic)

Local Model Checking Problem

- Fixed a (Kripke) model M (a transition system), an initial state s_0, and a temporal property φ

$$M, s_0 \models \varphi?$$

where $\models$ = satisfiability relation

Global Model Checking Problem

- Fixed a (Kripke) model M and a temporal property φ, compute all states s such that $M, s \models \varphi$.
- (Global solves Local)

Linear Temporal Logic

PLTL Syntax

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- Classical connectives:

$$\neg\psi \quad \varphi \wedge \psi \quad \varphi \vee \psi \quad \varphi \supset \psi$$

Temporal Operators

A formula withot modalities at the top level is evaluated in the current instant

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- $\mathbf{G}\varphi$: globally/always φ
- $\varphi\mathbf{U}\psi$: φ until ψ

Logical equivalences

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- Indeed,

$$\begin{aligned} \mathbf{F}\varphi &\equiv \mathit{true}\ \mathbf{U}\ \varphi \\ \mathbf{G}\varphi &= \neg\mathbf{F}\neg\varphi \end{aligned}$$

Traffic Light

- once green, next state is not red

$$\mathbf{G}(\mathit{green} \supset \neg \mathbf{X}\,\mathit{red})$$

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$$\mathbf{F}\ \mathit{green}$$

- once green, becomes red after being yellow for some time

$$\mathbf{G}(\mathit{green} \supset ((\mathit{green}\ \mathbf{U}\ \mathit{yellow})\ \mathbf{U}\ \mathit{red}))$$

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  - We can represent it as a finite graph (labels over 2^{AP}!)

Satisfiability

- For $\sigma = \langle S, R, L \rangle$, let $R^j(s) = \underbrace{R(\ldots R(s)\ldots)}_{j}$

  The relation $\sigma, s \models \varphi$ (σ satisfies φ in s) is defined as

  - $\sigma, s \models p$ if $p \in L(s)$

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  - $\sigma, s \models \varphi_1 \; \mathbf{U} \; \varphi_2$ if
    $\exists j \geq 0.\ \sigma, R^j(s) \models \varphi_2,\ (\forall 0 \leq k < j.\ \sigma, R^k(s) \models \varphi_1)$

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  - $\sigma, s \models \mathbf{F}\varphi$ if $\exists j.\ \sigma, R^j(s) \models \varphi$
  - $\sigma, s \models \mathbf{G}\varphi$ if $\forall j.\ \sigma, R^j(s) \models \varphi$

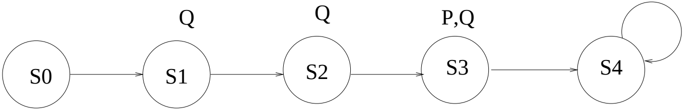
Example 1

- If $p\mathbf{U}q$ holds in s, then $\mathbf{F}q$ holds in s,
- The weak until operator:

$$p\ \mathbf{W}\ q \ \equiv\ \mathbf{G}p\ \vee\ (p\ \mathbf{U}\ q)$$

Example 2

Let M be the following model



$L(s_0) = \emptyset$, $L(s_1) = \{q\}$, $L(s_2) = \{q\}$, $L(s_3) = \{p, q\}$, $L(s_4) = \emptyset$ then

- $\mathbf{X}p$ is true in s_2

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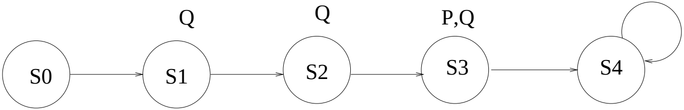

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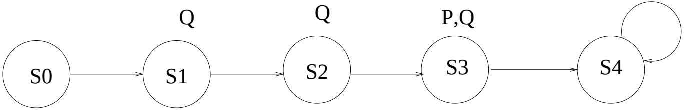

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- $\mathbf{X}p$ is true in s_2
- $\mathbf{F}p$ is true in s_0, s_1, s_2, s_3
- $\mathbf{G}p$ is never true

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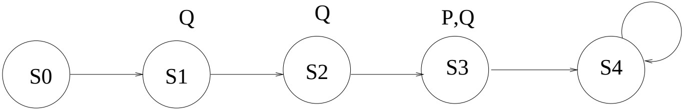

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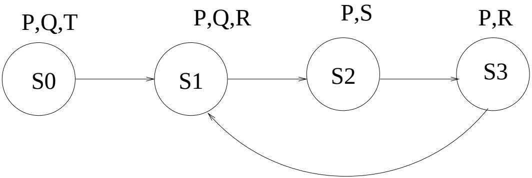
then

- $\mathbf{X}p$ is true in s_2
- $\mathbf{F}p$ is true in s_0, s_1, s_2, s_3
- $\mathbf{G}p$ is never true
- $q\ \mathbf{U}\ p$ is true in s_1, s_2, s_3

Example II

- Let M be the following model



- then
  - $\mathbf{F}t$ is true in s_0

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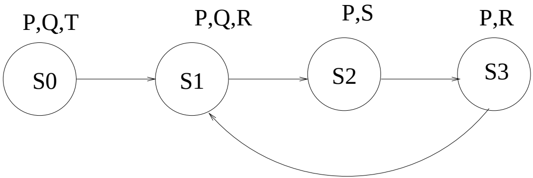

- then
  - $\mathbf{F}t$ is true in s_0
  - $\mathbf{G}p$ is true in all states

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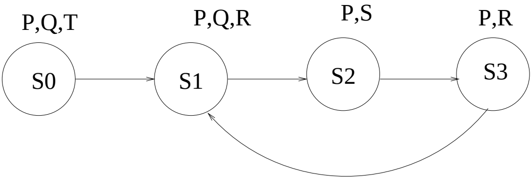

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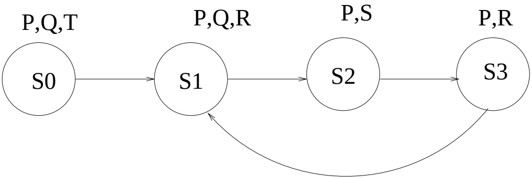

- then
  - $\mathbf{F}t$ is true in s_0
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  - $\mathbf{G}\ \mathbf{F}s$ is true in all states
  - $\mathbf{X}(r \supset (q\ \mathbf{U}\ s))$ is true in s_0, s_1, s_3

Interesting LTL formulas

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- $\mathbf{F}\ \mathbf{G}p$: when p becomes true, it remains true forever (eventually permanently)
- $\mathbf{G}(p \supset \mathbf{F}q)$: globally, if p holds then eventually q holds (responsiveness)

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Axioms for LTL: Duality

$$\neg \mathbf{G}\varphi \equiv \mathbf{F}\neg\varphi$$

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$$\neg \mathbf{X}\varphi \equiv \mathbf{X}\neg\varphi$$

Axioms for LTL: Expansion

$$\varphi \,\mathbf{U}\, \psi \;\equiv\; \psi \;\vee\; [\varphi \;\wedge\; \mathbf{X}(\varphi \,\mathbf{U}\, \psi)]$$

$$\mathbf{F}\varphi \;\equiv\; \varphi \;\vee\; \mathbf{X}\;\mathbf{F}\varphi$$

$$\mathbf{G}\varphi \;\equiv\; \varphi \;\wedge\; \mathbf{X}\;\mathbf{G}\varphi$$

Axioms for LTL: Idempotence

$$\mathbf{G}\ \mathbf{G}\varphi \equiv \mathbf{G}\varphi$$

$$\mathbf{F}\ \mathbf{F}\varphi \equiv \mathbf{F}\varphi$$

$$\varphi\ \mathbf{U}\ (\varphi \mathbf{U} \psi) \equiv \varphi\ \mathbf{U}\ \psi$$

$$(\varphi\ \mathbf{U}\ \psi)\mathbf{U}\psi \equiv \varphi\ \mathbf{U}\ \psi$$

Axioms for LTL: Absorbtion

$$\mathbf{F}\ \mathbf{G}\ \mathbf{F}\varphi \equiv \mathbf{G}\ \mathbf{F}\varphi$$

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Axioms for LTL: Commutation

$$\mathbf{X}(\varphi \ \mathbf{U} \ \psi) \quad \equiv \quad (\mathbf{X}\varphi) \ \mathbf{U} \ (\mathbf{X} \ \psi)$$

A Deduction System for LTL

Axioms:

$$
\begin{array}{l}
\textit{All tautologies} \\
\mathbf{X}(\neg\varphi) \equiv \neg\mathbf{X}\varphi \\
\mathbf{X}(\varphi \supset \psi) \supset \mathbf{X}\varphi \supset \mathbf{X}\psi \\
\mathbf{G}\varphi \supset (\varphi \wedge \mathbf{X}\mathbf{G}\varphi)
\end{array}
$$

Rules:

Modus ponens	$\varphi, \varphi \supset \psi \vdash \psi$
Next	$\varphi \vdash \mathbf{X}\varphi$
Indution	$\varphi \supset \psi, \varphi \supset \mathbf{X}\varphi \vdash \varphi \supset \mathbf{G}\psi$

Derivation

$$(\mathbf{X}A \supset \mathbf{X}B) \supset \mathbf{X}(A \supset B)$$

- $(\neg(A \supset B)) \supset A$ [taut]

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Derivation

$$(\mathbf{X}A \supset \mathbf{X}B) \supset \mathbf{X}(A \supset B)$$

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Other Theorem

$$(A \wedge \mathbf{XF}A) \supset \mathbf{F}A$$

- ...

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- No, it is false if A is true in s_1 and false in s_3!
  Actually, it can not be expressed in LTL

Lemma

Assumptions

- Let $M(i) = p^i(\neg p)p^\omega$ be the model in which p is true in $s_0 s_1 \ldots s_i$, false in state s_{i+1}, and true in s_j for $j > i + 1$.
- $M(0), M(2), M(4), \ldots$ all satisfy $even(p)$, while $M(1), M(3), \ldots$ don't

Lemma

- Let φ be a formula on predicate p with k occurrences of $\mathbf{X}$

Property
For $i > k$, the truth of φ on $M(i)$ is independent from i
i.e. φ has always the same value in $p^{k+1}(\neg p)p^{\omega}$, $p^{k+2}(\neg p)p^{\omega}, \ldots$
P. Wolper. Temporal Logic can be more expressive

Proof of the Lemma

By induction on the structure of φ.
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- etc.

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How to express $even(p)$?

- Automata: that recognizes $p, true, p, true, \ldots$ (infinite word)
- ETL (Wolper): We can express $even(p)$ by quantifying over predicates:

$$\exists q.(q \wedge \mathbf{G}(q \supset \mathbf{X}\neg q) \wedge \mathbf{G}(\neg q \supset \mathbf{X}q) \wedge \mathbf{G}(q \supset p))$$

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Let us consider models with several infinite paths.

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- Existential Model checking: Fixed a model M, a state s_0 and an LTL formula φ is there an infinite path σ in M s.t. $\sigma, s_0 \models \varphi$? (dual to the first problem)

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- The problems are decidable but with exponential-time in the size of φ

Model Checking Algorithm

- Tableau methods

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Semantic Tableau in Classical Logic

(See slides on Aulaweb)

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(See slides on Aulaweb)

- Deductive method for checking a formula
- Goal: build a model using syntactic rules (semantic method)

LTL Model Checking Algorithm

Lichtenstein-Pnueli's Algorithm: Tableau

- For fixed M, s, φ, decide $M, s \models \psi$
- The algorithm is based on the construction of a tableau, i.e., the product of M with a syntactic model for the formula $\varphi = \neg\psi$ (solve existential model checking)
- The states of the tableau contains sets of subformulas of φ (Hintikka sets) from the closure of φ
- We restrict the construction to formulas with **X** and **U**

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- The cardinality of $cl(\varphi)$ is linear in φ

Example: Closure of a formula

$cl(p_1 \mathbf{U} p_2)$
contains
$p_1 \mathbf{U} p_2$,
p_1,
p_2,
$\mathbf{X}(p_1 \mathbf{U} p_2)$
and all negated formulas

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- (logically consistent) for each $\psi_1 \vee \psi_2 \in cl(\varphi)$, $\psi_1 \vee \psi_2 \in X$ iff $\psi_1 \in X$ or $\psi_2 \in X$
- (locally consistent) for each $\psi_1 \ \mathbf{U}\ \psi_2 \in cl(\varphi)$, $\psi_1 \ \mathbf{U}\ \psi_2 \in X$ iff either $\psi_2 \in X$, or $(\{\psi_1, \mathbf{X}(\psi_1 \ \mathbf{U}\ \psi_2)\} \subseteq X)$

Tableau

Fixed a model M:

- Nodes = atoms of the form $\langle s_A, K_A \rangle$ where s_A is a state in M and K_A is a maximal consistent set compatible with the labelling of s_A
- Edge = From $\langle s_A, K_A \rangle \longrightarrow \langle s_B, K_B \rangle$ iff
  - $R(s_A) = s_B$ is a transition in M
  - For each formula $\mathbf{X}\varphi_1 \in CL(\varphi)$, $\mathbf{X}\varphi_1 \in K_A$ iff $\varphi_1 \in K_B$

Strongly Connected Component

A strongly connected component of a graph is a subgraph C s.t. for each pair $\langle n, n' \rangle$ of nodes in C there exists a path from n to n'

Self-fulfilling

A strongly connected component C is self-fulfilling iff:
for each atom B in C and for each formula $\varphi_1 \mathbf{U} \varphi_2 \in B$,
there exists an atom B' in C such that $\varphi_2 \in K_{B'}$

Theorem [Pnueli-Lichtenstein]

- Let G be the tableau associated to M and ψ
- $M, s \models_{\exists} \psi$ iff:
    - there exists an atom $A = \langle s, K \rangle$ in G s.t. $\psi \in K$
    - there exists a path in G from A to a self-fulfilling strongly connected component C of G

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- We can use again a DFS for checking reachability of a SFSCC

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  - the problem is the size of M, it can be exponential in its description!
- The two DFS visits can be nested, and the search of an accepting state can be made on-the-fly (we build the tableau while searching for a lasso)

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- The tableau algorithm can be viewed in terms of automata operation: product and emptiness test
- Automata theory allows to exploit optimal algorithms for such operations
- We need special automata that accept infinite-words